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Hecke Algebras and Knot Polynomials— Finding a Link Invariant

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1 Introduction

Knot invariants before Hecke algebras

A knot is a continuous embedding of a circle S^1 into 3-dimensional Euclidean space $\mathbb{R}^3$ (or into the 3-Sphere S^3) without self-intersection and a link is a finite set of non-intersecting knots which may or may not be linked together. Two knots (or links) are said to be isotopic, if one can be continuously transformed into the other without self-intersection. The process of deciding whether two knots are non-isotopic has been a central problem in low dimensional topology. The search for isotopy invariants has led to advancements in topology and representation theory. In this project, we will tackle one those isotopy invariants by representing a link as an element of the braid group and analyzing the associated Hecke algebra of the latter.

Artin [2] showed that the set of braids built from n strands is a group with a common presentation

$$B_n = \left\langle \sigma_1, \dots, \sigma_{n-1} \mid \begin{array}{ll} \sigma_i \sigma_j = \sigma_j \sigma_i & |i - j| > 2 \\ \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} & i \in \{1, \dots, n-2\} \end{array} \right\rangle.$$

In fact, every link (in S^3) can be isotopically represented as the closure of some braid [1]. This allows us to study links algebraically using the representations of braid groups. Markov [8] showed that two braids represent the same link (via the closure operation) if and only if they differ by a sequence of *Markov moves* [8].

$$\begin{aligned} \alpha &\mapsto \beta \alpha \beta^{-1} & \alpha, \beta &\in B_n \\ \alpha &\mapsto \alpha \sigma_n^{\pm 1} & \alpha &\in B_n. \end{aligned}$$

This equivalence relation gives us a way to classify links using the braid group B_n .

Jones' approach

This work will mainly follow the approach taken in Jones [6]. He discovered a new polynomial knot invariant and was awarded the Fields Medal in 1990.

By introducing another quadratic relation for some parameter $q \in \mathbb{C}$

$$\sigma_i^2 = (q - 1)\sigma_i + q$$

we obtain the so called Hecke algebra of type A_{n-1}

$$H_n(q) := \mathbb{C}[B_n] / \sigma_i^2 = (q - 1)\sigma_i + q.$$

In fact, there are several reasons for introducing this quadratic relation. Jones [6] showed that it is closely related to the *skein relations*, which uniquely define the two-variable polynomial $P_L(x, t)$ called the *HOMFLY-PT* polynomial named after Hoste, Ocneanu, Millett, Freyd, Lickorish and Yetter who discovered it in [4] and Przytycki and Traczyk who discovered it independently in [9]. The HOMFLY-PT polynomial is uniquely defined by the skein rule, that is if (isotopy classes of oriented) L_+ , L_- , L_0 are links with the same projection except for one crossing as in figure 1, then $t^{-1}P_{L_+} - tP_{L_-} = xP_{L_0}$ (from Jones [6]). This polynomial coincides with the Jones invariant for some choice of variables. This gives us a topological reason for introducing this quadratic relation.

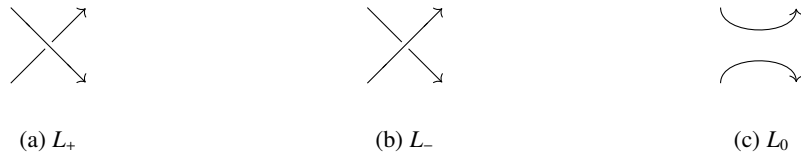


Figure 1: skein relations

Algebraically, the relation makes the Hecke algebra a q -deformation of the group algebra $\mathbb{C}[S_n]$ of the symmetric group S_n and as such, it shares some nice properties of $\mathbb{C}[S_n]$, in particular semi-simplicity (at least for good parameters q) and the correspondence between its irreducible representations and Young tableaux (see theorems 3.3 and 4.1). By defining a trace function on this Hecke algebra, respecting Markov moves, it is possible to construct Jones' polynomial link invariant. One focus of this work will be to analyze explicitly an alternative way to calculate this trace (at least for low n) using characters of irreducible representations.

Link invariant in a solid torus

Towards an algebraic generalization, one builds an Artin group and a Hecke algebra out of every Coxeter group. In the case of S^3 , the regular braid group B_n is the Artin group of the Coxeter group S_n and $H_n(q)$ its Hecke algebra. For the Coxeter group $C_2^n \rtimes S_n$, one has a topological interpretation as its Artin group is the braid group $B_{1,n}$ of the solid 2-torus. Similarly to S^3 , this gives rise to a polynomial link invariant for links in a solid torus as discussed in Lambropoulou [7]. In the solid torus, the braid group $B_{1,n}$ has a common presentation

$$B_{1,n} = \left\langle T, \sigma_1, \dots, \sigma_{n-1} \left| \begin{array}{ll} \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} & \forall i \\ \sigma_i \sigma_j = \sigma_j \sigma_i & |i - j| \geq 2 \\ T \sigma_i = \sigma_i T & i \geq 2 \\ T \sigma_1 T \sigma_1 = \sigma_1 T \sigma_1 T \end{array} \right. \right\rangle$$

This gives rise to the Hecke algebra of B_n -type $H_n(q, Q)$ with a trace function, which in turn gives rise to a different polynomial invariant for the solid torus. There will be a short discussion of the relation between the two polynomial invariants, and it will be shown that the one in the solid torus does not specialize by elementary means to the one in S^3 (see end of section 5.1).

Breakdown and further studies

In this project, we will follow Jones' approach by introducing the Hecke algebra in section 4.1 and defining a trace on it (section 4.2). We will look at multiple ways to calculate this trace (section 4.3) and end Jones' discussion with the construction of the link invariant in section 4.4. In section 5, we will follow Lambropoulou [7] using a similar approach to find a link invariant for the solid 2-torus instead of S^3 .

The content of this project suggests readily further generalizations which we would have liked to explore if time permitted. In particular, it would be interesting to see, whether other Coxeter groups (or more precisely their Artin groups) correspond to braid groups in other 3-manifolds and whether it is also possible to construct a polynomial invariant by using their corresponding Hecke algebras.

2 Acknowledgements

I would like to thank Eric Chen for guiding me through this project and addressing any questions or concerns that arose. Also, I would like to thank all my lecturers at EPFL who inspired me to continue pursuing mathematics, and to keep asking questions. Lastly, I want to thank my family and friends for their support and the occasional proofreading, especially my father who, as a non-mathematician, had the pleasure of looking at section A.1 and even managed to find several typos.

3 Algebraic Preliminaries

We begin by looking at some important notions of representation theory to be able to tackle our goal of finding a link invariant. We will use semisimplicity a lot, so let us define it.

Definition 3.1 (Semisimplicity, Etingof et al. [3, Definition 3.5.7]). *A finite-dimensional algebra A is semisimple if its radical $\text{rad}(A) = 0$, i.e. it does not have elements that act by 0 in all irreducible representations. Equivalently, every A -module splits as a direct sum of irreducible representations.*

Theorem 3.1 (Fulton and Harris [5, Chapter 3.4]). *Let G be a finite group. Then $\mathbb{C}[G]$ is semisimple.*

Theorem 3.2 (Wedderburn—Artin, Etingof et al. [3, Proposition 3.5.8]). *For any finite-dimensional semisimple algebra A , we have*

$$A \simeq \bigoplus_i \text{Mat}_{d_i}(k)$$

for some d_i

This will be used in section 4.3 to provide an alternative approach to calculating the trace on the Hecke algebra $H_n(q)$. For constructing representations of S_n , we need the following:

Definition 3.2 (Restriction, Etingof et al. [3, Chapter 5.8]). *Given a representation V of a finite group G and a subgroup $H \subset G$, the restriction $\text{Res}_H^G V$ is the representation on H given by $\rho = \rho_V|_H$.*

Definition 3.3 (Induced representation, Etingof et al. [3, Definition 5.8.1]). *Let G be a group and $H \subset G$ a subgroup and V a representation of H (over a field k). Then the induced representation $\text{Ind}_H^G V$ is the representation of G with*

$$\text{Ind}_H^G V = \{f : G \rightarrow V \mid f(hx) = \rho_V(h)f(x) \forall x \in G, h \in H\}$$

and the action $g(f)(x) = f(xg) \forall g \in G$. More generally, any k -linear representation V of H

$$\text{Ind}_H^G V = k[G] \otimes_{k[H]} V$$

Definition 3.4 (Young diagram, Etingof et al. [3, Chapter 5.12]). *A partition λ of n is a representation of n in the form*

$$n = \lambda_1 + \dots + \lambda_k$$

where $\lambda_i \in \mathbb{N}$ and $\lambda_i \geq \lambda_{i+1}$ for all i . For each partition λ , we can associate a Young diagram which is a grid with p rows where row i contains λ_i rectangles.

Example 3.1. For example, for $10 = 5 + 3 + 2$, the associated Young diagram is

$$Y = \begin{array}{|c|c|c|c|c|} \hline & & & & \\ \hline & & & & \\ \hline & & & & \\ \hline & & & & \\ \hline \end{array}$$

Theorem 3.3 (Etingof et al. [3, Chapters 5.12–5.16]). *There is a bijection between Young diagrams uniquely determined by theorem 3.4 and by identifying the trivial representation $S_n \xrightarrow{x \rightarrow 1} \mathbb{C}$ with the partition $\lambda = n$, which has a Young diagram being a row of n boxes.*

Theorem 3.4 (Etingof et al. [3, Chapter 5.16]). *For a partition λ , $\text{Res}_{S_{n-1}}^{S_n} V_\lambda$ is isomorphic to the direct sum of representations corresponding to the Young diagrams obtained by removing a square from λ . $\text{Ind}_{S_{n-1}}^{S_n} V_\lambda$ is isomorphic to the direct sum of representations corresponding to the Young diagrams obtained by adding a square to λ .*

This allows us to build representations of S_n inductively from representations for lower n , which we will use extensively in section 4.3. Though not used explicitly, it is nice to know about Coxeter groups and their Artin groups to understand the general procedure we are following.

Definition 3.5 (Coxeter group, Lambropoulou [7]). *A group with presentation*

$$G = \langle w_1, \dots, w_n \mid (w_i w_j)^{m_{ij}}, m_{ii} = 1 \forall i \rangle$$

where $m_{ij} \geq 2$ is an integer or infinity (in which case, no relation is imposed on the two generators).

Definition 3.6 (Artin group, Lambropoulou [7]). *For a Coxeter group G as above, its Artin group is given by*

$$\left\langle \tau_1, \dots, \tau_n \mid \underbrace{\tau_i \tau_j \tau_i \dots}_{m_{ij} \text{ terms}} = \underbrace{\tau_j \tau_i \tau_j \dots}_{m_{ij} \text{ terms}} \right\rangle$$

4 The Jones polynomial

To study links, it is useful to study them algebraically and not just using knot/link-projection diagrams. To this end, we can use the braid group B_n which consists of n strands tangled with each other. A common presentation given by Artin [2] is

$$B_n = \left\langle \sigma_1, \dots, \sigma_{n-1} \left| \begin{array}{ll} \sigma_i \sigma_j = \sigma_j \sigma_i & |i - j| > 2 \\ \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} & i \in \{1, \dots, n-2\} \end{array} \right. \right\rangle. \quad (1)$$

Pictorially, the generator σ_i is given by figure 2.

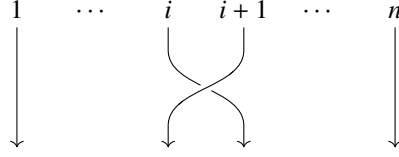


Figure 2: Generator σ_i

The first relation in equation (1) corresponds to two generators commuting if they are sufficiently far apart. One can convince oneself that the second relation also makes sense intuitively, see figure 3.

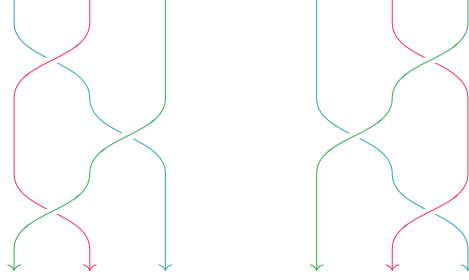


Figure 3: Relation $\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$

For a braid $\alpha \in B_n$, we can consider its closure $\hat{\alpha}$ —by connecting the ends of the strands with the starts (see figure 4)—which gives us a link diagram. Indeed, every link can be written as the closure of some braid $\alpha \in B_n$ according to Alexander [1]. Markov [8] showed that two links are isotopic if and only if they differ by a finite number of *Markov Moves*. This gives us an equivalence relation on $\bigcup_{n \geq 1} B_n$ given by equation (2). A pictorial version of this is given in figure 5.

$$\alpha \sim \beta \alpha \beta^{-1} \quad \alpha, \beta \in B_n \quad (2a)$$

$$\alpha \sim \alpha \sigma_n^{\pm 1} \quad \alpha \in B_n \quad (2b)$$

Our goal now is to find a link invariant that can be calculated from the braid representation of a link. As such, we need to respect the *Markov Moves* in the definition of our invariant.

4.1 A Hecke algebra

A convenient way to find our invariant is to introduce the notion of a Hecke Algebra.

Remark 4.1. For simplicity, we will only define the Hecke algebra of the braid group B_n but this can be generalized (see section 5).

Given the representation of the braid group and a complex number $q \in \mathbb{C}$, we can define the Hecke algebra $H_n(q)$ by adding a quadratic relation for the generators to B_n , given in equation (3).

$$H_n(q) := \left\langle g_1, \dots, g_{n-1} \left| \begin{array}{ll} g_i g_{i+1} g_i = g_{i+1} g_i g_{i+1} & i \in \{1, \dots, n-2\} \\ g_i g_j = g_j g_i & |i - j| > 2 \\ g_i^2 = (q-1)g_i + q & i \in \{1, \dots, n-1\} \end{array} \right. \right\rangle \quad (3)$$

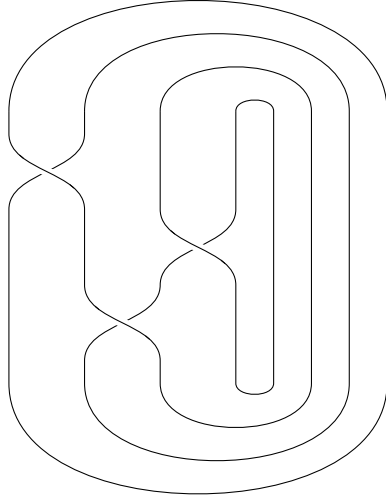
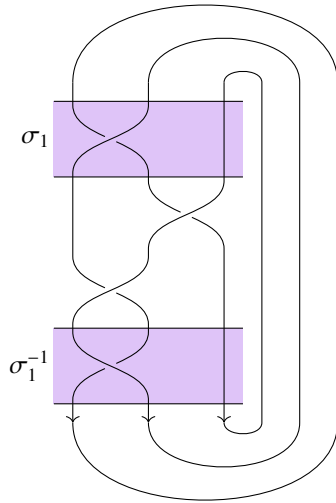
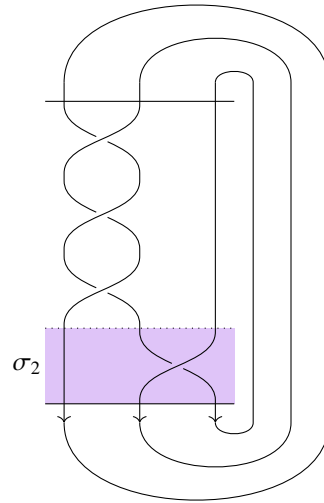


Figure 4: Closure of the braid $\sigma_1 \sigma_3 \sigma_2$



(a) Markov move of type I:
 $\sigma_1 \alpha \sigma_1^{-1}$ with $\alpha = \sigma_2 \sigma_1$



(b) Markov move of type II:
 $\alpha \sigma_2$ with $\alpha = \sigma_1^3$

Figure 5: Markov Moves

Remark 4.2. Observe that this is a quotient of the group algebra of the braid group $H_n(q) \simeq \mathbb{C}[B_n]/\langle \sigma_i^2 = (q-1)\sigma_i + q \rangle_{i=1}^{n-1}$.

For future use, we make note of the following computation:

Lemma 4.1. We have

$$g_i^{-1} = \frac{g_i - q + 1}{q} \quad (4)$$

From the relation $g_i^2 = (q-1)g_i + q$, we can deduce the following linear recurrence relation (for $k \geq 2$)

$$\begin{aligned} g_i^k &= g_i^{k-2} g_i^2 \\ &= g_i^{k-2} ((q-1)g_i + q) \\ &= (q-1)g_i^{k-1} + qg_i^{k-2} \end{aligned} \quad (5)$$

with solution given in section A.2.

Note that if we set $q = 1$, we obtain the relation $g_i^2 = 1$, which implies that $H_n(1) = \mathbb{C}[S_n]$. As such, $H_n(q)$ is a q -deformation of S_n and inherits some of its nice properties:

Lemma 4.2 (Jones [6]). $H_n(q)$ has a basis

$$\{(g_{i_1} \dots g_{i_1-k_1}) \dots (g_{i_p} \dots g_{i_p-k_p}) \mid 1 \leq i_1 < \dots < i_p \leq n-1\} \quad (6)$$

Theorem 4.1 (Jones [6]). For $q = 1$ or q not a root of unity, $H_n(q)$ is semisimple. Its modules are in one-to-one correspondence with Young diagrams of n boxes. Their composition rules are the same. The restriction of a representation of $H_n(q)$ to $H_{n-1}(q)$ corresponding to a Young diagram of n boxes is isomorphic to the direct sum of representations of $H_{n-1}(q)$ obtained by removing one box of said Young diagram obtaining Young diagrams with $n-1$ boxes. Therefore, their dimensions are also the same.

Example 4.1 (Illustration of theorem 4.1). For example the restriction of the following irreps of $H_9(q)$ to $H_8(q)$ composes into the direct sum

$$\text{Res}_{H_8(q)}^{H_9(q)} \begin{array}{|c|c|c|c|c|} \hline & & & & \\ \hline & & & & \\ \hline & & & & \\ \hline & & & & \\ \hline & & & & \\ \hline \end{array} = \begin{array}{|c|c|c|c|} \hline & & & \\ \hline & & & \\ \hline & & & \\ \hline & & & \\ \hline & & & \\ \hline \end{array} \oplus \begin{array}{|c|c|c|c|c|} \hline & & & & \\ \hline & & & & \\ \hline & & & & \\ \hline & & & & \\ \hline & & & & \\ \hline \end{array} \oplus \begin{array}{|c|c|c|c|} \hline & & & \\ \hline & & & \\ \hline & & & \\ \hline & & & \\ \hline & & & \\ \hline \end{array}.$$

We will use this fact later, to construct the trace from section 4.2 using irreducible representations.

4.2 Ocneanu's Trace function

We are looking to find a linear trace function $\text{tr} : \bigcup_{n \geq 1} H_n(q) \rightarrow \mathbb{C}$ such that the Markov moves are respected. The existence of such a trace function was discovered as a consequence of the following theorem from Jones [6].

Theorem 4.2 (Jones [6]). For every $z \in \mathbb{C}$ there exists a linear trace function

$$\text{tr} : \mathcal{H}(q) := \bigcup_{n \geq 1} H_n(q) \rightarrow \mathbb{C}$$

uniquely defined by

$$\text{tr}(\alpha\beta) = \text{tr}(\beta\alpha) \quad \forall \alpha, \beta \quad (7a)$$

$$\text{tr}(ag_n) = z \text{tr}(\alpha) \quad \alpha \in H_n(q) \quad (7b)$$

$$\text{tr}(1) = 1 \quad (7c)$$

$\forall$

Proof of theorem 4.2 adapted from Jones [6]. Within this proof, we will write H_n for $H_n(q)$ for simplicity. First, let us construct an isomorphism

$$\begin{aligned} C : H_n \oplus H_n \otimes_{H_{n-1}} H_n &\rightarrow H_{n+1} \\ (x, y_1 \otimes y_2) &\mapsto x + y_1 g_n y_2 \end{aligned}$$

Note that the generator g_n can appear at most once in a basis element of the from equation (6) (note also, that we replace $n - 1$ with n). Thus, we may write a basis element in H_{n+1} of the form $y_1 g_n y_2$ for $y_1, y_2 \in H_n$. So, clearly C is surjective: $C(0, y_1 \otimes y_2) = y_1 g_n y_2$.

The injectivity of C follows by counting the number of dimensions. The algebra H_n is a free H_{n-1} -module of rank n . And so, its dimension is $n \times (n - 1)! = n!$. Furthermore, we have the isomorphism $H_n \otimes_{H_{n-1}} H_n \simeq \oplus_{n=1}^n H_n$. Finally, we calculate

$$\begin{aligned} \dim(H_n \oplus H_n \otimes_{H_{n-1}} H_n) &= n! + n \times n! \\ &= (n + 1)! \end{aligned}$$

So C is indeed an isomorphism.

Now, we can define the trace recursively by

$$\begin{aligned} \text{tr}(1) &= 1 \\ \text{tr}(x g_n y) &= z \text{tr}(xy) \quad \text{for } x, y \in H_n \end{aligned}$$

Now, all that is left to show is the symmetry. By induction, we can assume that $\text{tr}(xy) = \text{tr}(yx)$ for $x, y \in H_n$ and then prove it for $x, y \in H_{n+1}$. Note that this statement is trivial by using the induction hypothesis and the Markov condition whenever one of x and y does not contain g_n . We can further reduce elements of the form $x_1 g_n x_2 y_1 g_n y_2$ to the form $x g_n y g_n$ by using braid relations and combining terms.

Thus, we are left to prove that $\text{tr}(x g_n y g_n) = \text{tr}(y g_n x g_n)$.

Thus, we have the following possibilities:

1. If $x, y \in H_{n-1}$:

$$\begin{aligned} \text{tr}(x g_n y g_n) &= \text{tr}(x g_n^2 y) \\ &= (q - 1) \text{tr}(x g_n y) + q \text{tr}(xy) \\ &= (q - 1)z \text{tr}(xy) + q \text{tr}(xy) \end{aligned}$$

and

$$\begin{aligned} \text{tr}(y g_n x g_n) &= \dots \\ &= (q - 1)z \text{tr}(yx) + q \text{tr}(yx) \end{aligned}$$

2. If $x = a g_{n-1} b$ and $a, b, y \in H_{n-1}$:

$$\begin{aligned} \text{tr}(a g_{n-1} b g_n y g_n) &= \text{tr}(a g_{n-1} b g_n^2 y) \\ &= (q - 1) \text{tr}(a g_{n-1} b g_n y) + q \text{tr}(a g_{n-1} b y) \\ &= (q - 1)z \text{tr}(a g_{n-1} b y) + zq \text{tr}(a b y) \end{aligned}$$

and

$$\begin{aligned} \text{tr}(y g_n a g_{n-1} b g_n) &= \text{tr}(y a g_n g_{n-1} g_n b) \\ &= \text{tr}(y a g_{n-1} g_n g_{n-1} b) \\ &= z \text{tr}(y a g_{n-1}^2 b) \\ &= z(q - 1) \text{tr}(y a g_{n-1} b) + zq \text{tr}(y a b) \end{aligned}$$

3. The same as before with the roles of x and y switched

4. If $x = a g_{n-1} b, y = c g_{n-1} d$ for $a, b, c, d \in H_{n-1}$:

$$\begin{aligned} \text{tr}(a g_{n-1} b g_n c g_{n-1} d g_n) &= \text{tr}(a g_{n-1} b c g_n g_{n-1} g_n d) \\ &= \text{tr}(a g_{n-1} b c g_{n-1} g_n g_{n-1} d) \\ &= z \text{tr}(a g_{n-1} b c g_{n-1}^2 d) \\ &= z(q - 1) \text{tr}(a g_{n-1} b c g_{n-1} d) + zq \text{tr}(a g_{n-1} b c d) \\ &= z(q - 1) \text{tr}(a b c d g_{n-1}^2) + z^2 q \text{tr}(a b c d) \end{aligned}$$

and

$$\begin{aligned}
\text{tr}(cg_{n-1}dg_n ag_{n-1}bg_n) &= \text{tr}(cg_{n-1}dag_n g_{n-1}g_nb) \\
&= \text{tr}(cg_{n-1}dag_{n-1}g_n g_{n-1}b) \\
&= z \text{tr}(cg_{n-1}dag_{n-1}^2 b) \\
&= z(q-1) \text{tr}(cg_{n-1}dag_{n-1}b) + zq \text{tr}(cg_{n-1}dab) \\
&= z(q-1) \text{tr}(abcdg_{n-1}^2) + z^2q \text{tr}(abcd)
\end{aligned}$$

This concludes the proof. $\square$

Note that equation (7a) makes sure that the trace is invariant under Markov moves of type I (2a). Equation (7b) has a similar structure as the Markov moves of type II (2b).

4.3 Hecke algebra representations

Since $H_n(q)$ is semisimple and a q -deformation of S_n , it can be written as direct sum of matrix algebras indexed by Young diagrams

$$H_n(q) = \bigoplus_{\lambda} V_{Y_\lambda}.$$

The trace is determined by its restrictions on those component matrix algebras and the trace on a matrix algebra is necessarily a multiple of the usual trace. Thus, the trace can be written as a weighted sum traces of the irreducible representations of $H_n(q)$:

$$\text{tr} = \sum_{\lambda} w_{\lambda} \text{tr}_{\mathbb{C}^{n_{\lambda}}}$$

Jones [6] in section 5. gives a formula for those weights by considering the different Young diagrams. However, we will calculate them directly by computing the irreducible representations of $H_n(q)$ (for small n). For this, notice that the image of g_i in any one-dimensional representation φ of $H_n(q)$ must satisfy

$$\varphi(g_i)^2 = (q-1)\varphi(g_i) + q$$

and thus, g_i must get sent to either -1 or q . Furthermore, the relation $g_i g_{i+1} g_i = g_{i+1} g_i g_{i+1}$ imposes that all g_i map to the same value. This gives us the following lemma

Lemma 4.3. *The irreducible 1-dimensional representations of $H_n(q)$ are*

$$\begin{aligned}
\chi_{\text{triv}} : g_i &\mapsto q & \forall i \\
\chi_{\text{sgn}} : g_i &\mapsto -1 & \forall i
\end{aligned}$$

Example 4.2 ($H_2(q)$). *From the decomposition from theorem 4.1, we get the following character table*

	1	g_1
χ_{triv}	1	q
χ_{sgn}	1	-1

which gives us the following system of equations

$$\begin{cases} \text{tr}(1) = 1 = w_{\text{triv}} + w_{\text{sgn}} \\ \text{tr}(g_1) = z = w_{\text{triv}}q - w_{\text{sgn}}. \end{cases}$$

Solving this system gives $w_{\text{triv}} = (z+1)/(q+1)$ and $w_{\text{sgn}} = (q-z)/(q+1)$ and so for $\alpha \in H_2(q)$

$$\text{tr}(\alpha) = \frac{z+1}{q+1} \text{tr}_{\mathbb{C}}(\chi_{\text{triv}}(\alpha)) + \frac{q-z}{q+1} \text{tr}_{\mathbb{C}}(\chi_{\text{sgn}}(\alpha)).$$

Example 4.3 ($H_3(q)$). The irreps of $H_3(q)$ are:

$$\begin{aligned}\chi_{\text{triv}} : H_3 &\rightarrow \text{End}(\mathbb{C}) \\ g_i &\mapsto q \\ \chi_{\text{sgn}} : H_3 &\rightarrow \text{End}(\mathbb{C}) \\ g_i &\mapsto -1 \\ \rho : H_3 &\rightarrow \text{End}(\mathbb{C}^2) \\ g_1 &\mapsto \begin{pmatrix} -1 & 0 \\ 1 & q \end{pmatrix} \\ g_2 &\mapsto \begin{pmatrix} q & 1 \\ 0 & -1 \end{pmatrix}\end{aligned}$$

where ρ is the 2-dimensional irreducible representation corresponding to the Young diagram .

The trace evaluated at the basis elements of the algebra is

$$\begin{aligned}\text{tr}(1) &= 1 & \text{tr}(g_1) &= z \\ \text{tr}(g_2) &= z & \text{tr}(g_1 g_2) &= z^2 \\ \text{tr}(g_2 g_1) &= z^2 & \text{tr}(g_1 g_2 g_1) &= z^2(q-1) + zq\end{aligned}$$

Also, we have

H_3	$\xrightarrow{\psi}$	$\text{End}(\mathbb{C})$	$\oplus$	$\text{End}(\mathbb{C})$	$\oplus$	$\text{End}(\mathbb{C}^2)$	$\xrightarrow{a \text{tr} \oplus b \text{tr} \oplus c \text{tr}}$	$\mathbb{C}$
1	$\mapsto$	1	$\oplus$	1	$\oplus$	I_2	$\mapsto$	$a + b + 2c$
g_1	$\mapsto$	q	$\oplus$	-1	$\oplus$	$\begin{pmatrix} -1 & 0 \\ 1 & q \end{pmatrix}$	$\mapsto$	$aq - b + c(q-1)$
g_2	$\mapsto$	q	$\oplus$	-1	$\oplus$	$\begin{pmatrix} q & 1 \\ 0 & -1 \end{pmatrix}$	$\mapsto$	$aq - b + c(q-1)$
$g_1 g_2$	$\mapsto$	q^2	$\oplus$	1	$\oplus$	$\begin{pmatrix} -q & -1 \\ q & 1-q \end{pmatrix}$	$\mapsto$	$aq^2 + b + c(1-2q)$
$g_2 g_1$	$\mapsto$	q^2	$\oplus$	1	$\oplus$	$\begin{pmatrix} 1-q & q \\ -1 & -q \end{pmatrix}$	$\mapsto$	$aq^2 + b + c(1-2q)$
$g_1 g_2 g_1$	$\mapsto$	q^3	$\oplus$	-1	$\oplus$	$\begin{pmatrix} q-1 & -1 \\ 1-2q & q-q^2 \end{pmatrix}$	$\mapsto$	$aq^3 - b - c(q^2 - 2q + 1)$

where ψ is the isomorphism $H_3(q) \xrightarrow{\psi} \text{End}(\mathbb{C}) \oplus \text{End}(\mathbb{C}) \oplus \text{End}(\mathbb{C}^2)$ from theorem 4.1. Thus, a , b and c must satisfy

$$\begin{pmatrix} 1 & 1 & 2 \\ q & -1 & q-1 \\ q^2 & -1 & (1-2q) \\ q^3 & -1 & -(q-1)^2 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 1 \\ z \\ z^2 \\ z^2(q-1) + zq \end{pmatrix}$$

Solving this gives us the weights:

$$\begin{aligned}a &= \frac{(z+1)(qz+q+z)}{q(q+1)(q+2)} \\ b &= \frac{(-q+z)(-q^2+qz-q+z+1)}{q(q+1)(q+2)} \\ c &= -\frac{(-q+z)(z+1)}{q(q+2)}\end{aligned}$$

Finally, we can write the trace on $H_3(q)$ as

$$\text{tr}(\alpha) = a \text{tr}_{\mathbb{C}}(\chi_{\text{triv}}(\alpha)) + b \text{tr}_{\mathbb{C}}(\chi_{\text{sgn}}(\alpha)) + c \text{tr}_{\mathbb{C}^2}(\rho(\alpha))$$

As we can see, this process is quite tedious for bigger values of n . A formula to compute the weights of the traces directly is given in Jones [6].

4.4 The knot invariant

Equipped with the trace, we can finally look for our knot-invariant. For this, we need to normalize the trace in such a way that both Markov moves of type II (2b) affect the trace the same way, i.e. we want to find $\theta \in \mathbb{C}$ such that

$$\text{tr}(\theta g_i) = \text{tr}((\theta g_i)^{-1})$$

From there, we get

$$\begin{aligned} \lambda &:= \theta^2 = \text{tr}(g_i^{-1}) / \text{tr}(g_i) \\ &\stackrel{(4)}{=} \frac{1 - q + z}{qz} \end{aligned} \quad (8)$$

$$z = -\frac{1 - q}{1 - \lambda q} \quad (9)$$

$$\begin{aligned} \text{tr}(\lambda g_i) &= \text{tr}(\lambda g_i^{-1}) = z \sqrt{\lambda} \\ &= -\sqrt{\lambda} \frac{1 - q}{1 - \lambda q} \end{aligned} \quad (10)$$

Lemma 4.4 (Jones [6]). *By representing B_n with $\pi : \sigma_i \mapsto \sqrt{\lambda} g_i$, for $\alpha \in B_n$ the function*

$$(z \sqrt{\lambda})^{1-n} \text{tr}(\pi(\alpha))$$

is invariant under Markov moves.

Proof. For $\alpha \in B_n$, we have

$$\begin{aligned} (z \sqrt{\lambda})^{1-(n+1)} \text{tr}(\pi(\alpha \sigma_n^{\pm 1})) &= (z \sqrt{\lambda})^{-n} \text{tr}(\pi(\alpha) \sqrt{\lambda} g_n^{\pm 1}) \\ &= (z \sqrt{\lambda})^{-n} z \sqrt{\lambda} \text{tr}(\pi(\alpha)) \\ &= (z \sqrt{\lambda})^{1-n} \text{tr}(\pi(\alpha)) \end{aligned}$$

□

Finally, since we usually want to represent B_n by sending $\sigma_i \rightarrow g_i$, we correct the appearance of the $\sqrt{\lambda}$'s by introducing the exponent sum e on the generators σ_i . This finally gives us the two-variable polynomial invariant (11).

Theorem 4.3. *For a link $L = \hat{\alpha}$ where $\alpha \in B_n$, if we represent $\alpha \in B_n$ in $H_n(q)$ via $\pi : \sigma_i \rightarrow g_i$, the following two-variable (Laurent) polynomial in variables λ, z does not depend on the braid α and is thus a link invariant.*

$$X_L(\lambda, z) = z^{1-n} (\sqrt{\lambda})^{1-n+e} \text{tr}(\pi(\alpha)) \quad (11)$$

Example 4.4 (unknot). *Let us consider the unknot $L = \hat{1}$, where $1 \in B_1$. Applying equation (11) gives us*

$$X_L(z, \lambda) = z^{1-1} (\sqrt{\lambda})^{1-1+0} \text{tr}(1) = 1$$

The unknot can also be represented (via repeated usage of Markov moves of Type II) as the closure of $\sigma_1 \cdots \sigma_{n-1} \in B_n$. Then we have

$$\begin{aligned} X_L(z, \lambda) &= z^{1-n} (\sqrt{\lambda})^{1-n+(n-1)} \text{tr}(g_1 \cdots g_{n-1}) \\ &= z^{1-n} z^{n-1} = 1 \end{aligned}$$

So indeed, we get the same result for different representations of the link.

Example 4.5 (n -component unlink). *A bit more generally, we can consider the n -component unlink which is the set of n unknots which are not linked together. It can be represented as the closure of $1 \in B_n$. Then we get*

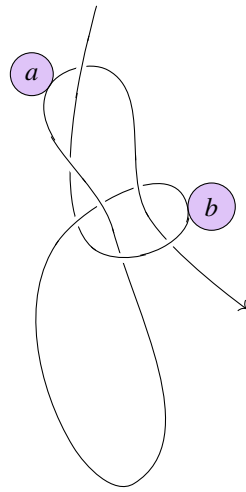
$$X_L(\lambda, z) = z^{1-n} (\sqrt{\lambda})^{1-n+0} \text{tr}(1) = (z \sqrt{\lambda})^{1-n}$$

Example 4.6 (bowline knot). *For fun, let us consider the bowline knot (see figure 6a) often used for sailing. One can convince oneself—by splitting the knot at a and b in figure 6—that it can be redrawn as the closure of the braid figure 6b.*

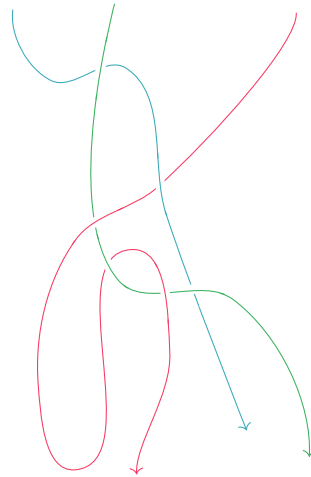
Thus, the bowline knot is the closure of the braid figure 7, or algebraically

$$\sigma_1 \sigma_2^{-1} \sigma_1^3 \sigma_2^{-1} \quad (12)$$

From there, we can calculate the two-variable knot polynomial shown in section A.1.



(a) Bowline Knot



(b) Bowline Knot as a braid

Figure 6: Reformulation of the bowline knot as a braid

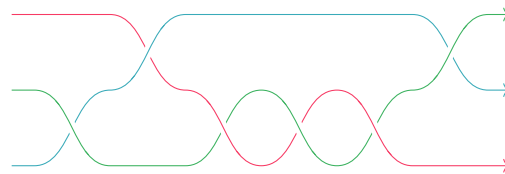


Figure 7: Bowline knot schematically

5 Generalizations

This chapter will follow Lambropoulou [7] in its effort to find a similar knot invariant for links in a solid torus (instead of usual links in S^3). Now as a topological space, we can view the solid torus M as the space S^3 minus another solid torus $\hat{I}$. This way, we can look at links in M as mixed links $\hat{I} \sqcup L$ consisting of a standard link L linking with the torus $\hat{I}$. Figure 8 visualizes those mixed links.

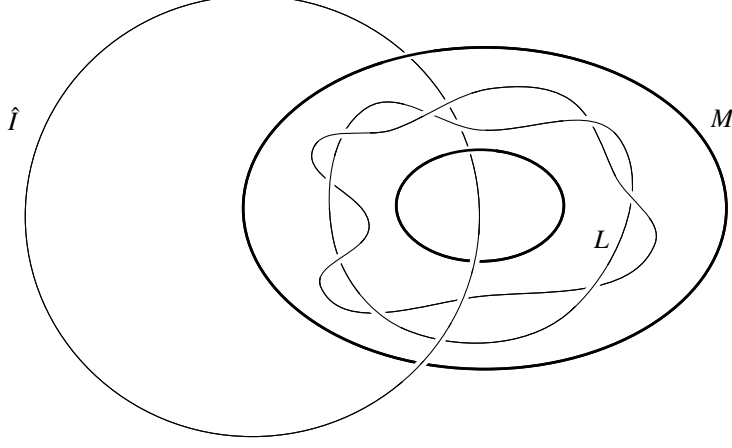


Figure 8: Link L in a torus M as $\hat{I} \sqcup L$

To be able to find an invariant polynomial analogous to equation (11) for a solid torus M , we encounter the following problem: When transforming a mixed link $\hat{I} \sqcup L$ into a braid, we need to make sure strand corresponding to the complementary torus $\hat{I}$ stays fixed (and in particular is not cut in the braiding process). Lambropoulou [7] showed that such a braiding algorithm does indeed exist. From there, we get to a version of Markov's theorem for a solid torus:

Theorem 5.1 (Lambropoulou [7]). *Two links containing the same braided part are isotopic if and only if any two corresponding braids, both containing the same braided part, differ by conjugation and, by Markov moves that do not affect the already braided part.*

We can represent any mixed link as the closure of a braid made of n normal strands and one special strand. This gives rise to a braid group $B_{1,n}$ with the following presentation:

$$B_{1,n} = \left\langle T, \sigma_1, \dots, \sigma_{n-1} \left| \begin{array}{ll} \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} & \forall i \\ \sigma_i \sigma_j = \sigma_j \sigma_i & |i - j| \geq 2 \\ T \sigma_i = \sigma_i T & i \geq 2 \\ T \sigma_1 T \sigma_1 = \sigma_1 T \sigma_1 T \end{array} \right. \right\rangle \quad (13)$$

The generator T corresponds to the first strand going around the special strand represented by figure 9a. The notation $T_i = g_1 \cdots g_i T g_i^{-1} \cdots g_1^{-1}$ (see figure 9b) will be useful for defining the trace in section 5.1. The relation $T \sigma_1 T \sigma_1 = \sigma_1 T \sigma_1 T$ is illustrated in figure 10 and can be seen by imagining sliding the upper loop through the lower to get from $T \sigma_1 T \sigma_1$ to $\sigma_1 T \sigma_1 T$.

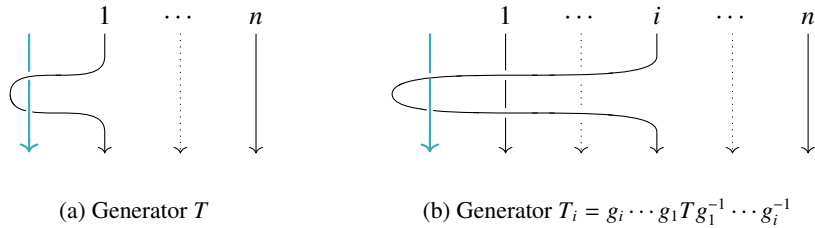


Figure 9: Generators T and T_i

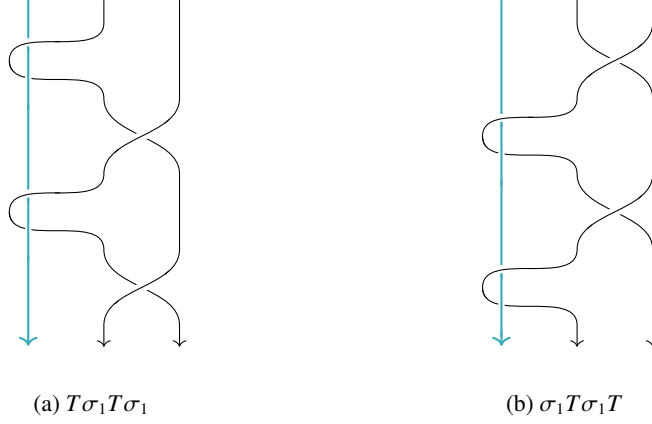


Figure 10: Relation $T\sigma_1 T\sigma_1 = \sigma_1 T\sigma_1 T$

5.1 A trace function

Similar to the approach taken in section 4.1, we can endow the braid group $B_{1,n}$ with additional relations to obtain a Hecke algebra suitable for the representing braids in a solid torus for two parameters $q, Q \in \mathbb{C}$:

$$H_n(q, Q) := \left\langle t, g_1, \dots, g_{n-1} \left| \begin{array}{ll} g_i g_{i+1} g_i = g_{i+1} g_i g_{i+1} & \forall i \\ g_i g_j = g_j g_i & |i - j| \geq 2 \\ t g_i = g_i t & i \geq 2 \\ t g_1 t g_1 = g_1 t g_1 t \\ g_i^2 = (q - 1)q_i + q & \forall i \\ t^2 = (Q - 1)t + Q \end{array} \right. \right\rangle \quad (14)$$

Again, we can define a trace on this Hecke algebra given by the following theorem:

Theorem 5.2 (Lambropoulou [7]). *Given $z, s \in \mathbb{C}$, there exists a linear trace function*

$$\text{tr} : \mathcal{H}(q, Q) := \bigcup_{n=1}^{\infty} H_n(q, Q) \rightarrow \mathbb{C}$$

uniquely defined by

$$\text{tr}(\alpha\beta) = \text{tr}(\beta\alpha) \quad \forall \alpha, \beta \quad (15a)$$

$$\text{tr}(\alpha g_n) = z \text{tr}(\alpha) \quad \alpha \in H_n(q, Q) \quad (15b)$$

$$\text{tr}(\alpha t_n) = s \text{tr}(\alpha) \quad \alpha \in H_n(q, Q) \quad (15c)$$

$$\text{tr}(1) = 1 \quad (15d)$$

where $t_n = g_n \cdots g_1 t g_1^{-1} \cdots g_n^{-1}$.

Proof. The proof is similar to the one in section 4.2 and is shown explicitly in Lambropoulou [7], as such we will not go through it here. $\square$

Example 5.1.

$$\begin{aligned} \text{tr}(t g_1 t g_1) &= \text{tr}(t g_1 g_1^{-1} g_1 t g_1^{-1} g_1 g_1) \\ &= \text{tr}(t t_1 g_1^2) \\ &= (q - 1) \text{tr}(t t_1 g_1) + q \text{tr}(t t_1) \\ &= (q - 1) \text{tr}(t g_1 t g_1^{-1} g_1) + q s \text{tr}(t) \\ &= (q - 1) \text{tr}(t^2 g_1) + q s^2 \\ &= (q - 1)(Q - 1) \text{tr}(t g_1) + (q - 1)Q \text{tr}(g_1) + q s^2 \\ &= (q - 1)(Q - 1) s z + (q - 1)Q z + q s^2 \end{aligned}$$

Remark 5.1. For calculating $\text{tr}(\alpha)$ for $\alpha \in H_n(q, Q)$ not containing any t_i 's, we only need equations (15a), (15b) and (15d) which are the same rules as for the trace on $H_n(q)$ (see theorem 4.2).

If we consider the embedding of the solid torus in Euclidean space $M \hookrightarrow S^3$, by just ignoring the extra strand, we get a map

$$\pi : H_n(q, Q) \xrightarrow{t \mapsto 1, Q \mapsto 1} H_n(q)$$

such that

$$\text{tr}_{\mathcal{H}(q, Q)} \circ \pi = \text{tr}_{\mathcal{H}(q)}$$

Note that considering the extra strand from $B_{1,n}$ as a normal strand in B_n would correspond to the group homomorphism

$$\begin{aligned} \psi : B_{1,n} &\rightarrow B_{n+1} \\ T &\mapsto \sigma_1^2 \\ \sigma_i &\mapsto \sigma_{i+1}. \end{aligned}$$

This is indeed a homomorphism. The only nontrivial thing to verify is that

$$\begin{aligned} tg_1tg_1 &\mapsto g_1^2g_2g_1^2g_2 \\ &= g_1g_1g_2g_1g_1g_2 \\ &= g_1g_2g_1g_2g_1g_2 \\ &= g_1g_2g_1g_1g_2g_1 \\ &= g_2g_1g_2g_1g_2g_1 \\ &= g_2g_1g_1g_2g_1g_1 \\ &= g_2g_1^2g_2g_1^2 \quad \leftarrow g_1tg_1t. \end{aligned}$$

But in fact, this group homomorphism does not extend to a Hecke algebra homomorphism $H_n(q, Q) \rightarrow H_{n+1}(q)$ shown by the following computation. First, we have

$$t \mapsto g_1^2 = (q-1)g_1 + q.$$

But then

$$\begin{aligned} t^2 &\mapsto g_1^4 \\ &\stackrel{(17)}{=} \frac{(q^4-1) + q + q^4}{q+1} \\ &= (q^2+1)(q-1)g_1 + q(q^2-q+1) \\ &= (q^2+1)((q-1)g_1 + q) - q^2 \end{aligned}$$

where on the other hand we have

$$\begin{aligned} t^2 &= (Q-1)t + Q \\ &\mapsto (Q-1)g_1^2 + Q \\ &= (Q-1)((q-1)g_1 + q) + Q. \end{aligned}$$

So, by comparing coefficients, we can see that Q cannot be specialized for generic q and so, there is no such canonical homomorphism of Hecke algebras. Thus, the trace in $\mathcal{H}(q, Q)$ does not specialize to the trace in $\mathcal{H}(q)$. In fact, this should not be surprising since we are essentially trying to look at a two-colored link (in $\mathcal{H}(q, Q)$) as a simple link.

5.2 Calculating the trace

The goal of this section is to find a way to calculate the trace explicitly by using usual traces of matrix algebras. In this project we do it for up to $n = 2$. First, let us remark that the Hecke algebra $H_n(q, Q)$ is built from the Coxeter group $W_n = 2^n \rtimes S_n$ as explained in Lambropoulou [7, Section 3.2]. For $n = 1$, this group is $C_2 \rtimes S_1 \simeq C_2$, the group with two elements with presentation $\langle a \mid a^2 = 1 \rangle$. The Hecke algebra $H_1(q, Q)$ is a (q, Q) -deformation of the group algebra $\mathbb{C}[C_2]$, so it is semisimple and there is a one-to-one correspondence of simple modules between the two algebras. Thus, we may write

$$H_1(q, Q) \simeq \text{End}(\mathbb{C}) \oplus \text{End}(\mathbb{C})$$

and so, we get the trace function as a weighted sum of usual trace functions, i.e. for $\alpha \in H_1(q, Q)$, we have

$$\text{tr}(\alpha) = a \text{tr}_{\mathbb{C}}(\rho_{\text{triv}}(\alpha)) + b \text{tr}_{\mathbb{C}}(\rho_{\text{sgn}}(\alpha)).$$

The irreducible representations of $H_1(q, Q) = \langle t \rangle$ are given by $\rho_{\text{triv}} : t \mapsto Q$ and $\rho_{\text{sgn}} : t \mapsto -1$ by the same argument as in lemma 4.3. This gives us the following system of equations where the first row corresponds to the trace of 1 and the second to the trace of t in the irreducible representations of $H_1(q, Q)$.

$$\begin{pmatrix} 1 & 1 \\ q & -1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 1 \\ s \end{pmatrix}$$

Solving this system gives us lemma 5.1.

Lemma 5.1. *The trace on $\text{tr} : H_1(q, Q) \rightarrow \mathbb{C}$ is given by*

$$\text{tr} = \frac{s+1}{q+1} \text{tr}_{\mathbb{C}} \circ \rho_{\text{triv}} + \frac{q-s}{q+1} \text{tr}_{\mathbb{C}} \circ \rho_{\text{sgn}}$$

For $n = 2$, the relevant Coxeter group is $C_2^2 \rtimes S_2$ which has a presentation

$$\langle a, b \mid (ab)^4 = 1, a^2 = b^2 = 1 \rangle \simeq D_4$$

which is isomorphic to the dihedral group of order 8 with the following irreducible representations:

$$\begin{aligned} \rho_0 : a &\mapsto 1 & b &\mapsto 1 \\ \rho_1 : a &\mapsto -1 & b &\mapsto -1 \\ \rho_2 : a &\mapsto 1 & b &\mapsto -1 \\ \rho_3 : a &\mapsto -1 & b &\mapsto 1 \\ \rho : a &\mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} & b &\mapsto \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \end{aligned}$$

Now, as before, $H_2(q, Q)$ is a q, Q -deformation of $\mathbb{C}[D_4]$ and so, its simple modules are in a one-to-one correspondence with the irreducible representations of D_4 . For $H_2(q, Q)$, the irreducible representations are

$$\begin{aligned} \rho_0 : t &\mapsto Q & g_1 &\mapsto q \\ \rho_1 : t &\mapsto -1 & g_1 &\mapsto -1 \\ \rho_2 : t &\mapsto Q & g_1 &\mapsto -1 \\ \rho_3 : t &\mapsto -1 & g_1 &\mapsto q \\ \rho : t &\mapsto \begin{pmatrix} Q & Q+q \\ 0 & -1 \end{pmatrix} & g_1 &\mapsto \begin{pmatrix} -1 & 0 \\ 1 & q \end{pmatrix} \end{aligned}$$

The two-dimensional representation is irreducible since the eigenvectors of $\rho(t)$ are $(q^{-1} \ 1)^T$ and $(0 \ 1)^T$ which are not invariant under $\rho(s)$, so there are no ρ -invariant subspaces. Also, one can easily verify that they satisfy the Hecke algebra relations, namely $\rho(t)^2 = (Q-1)\rho(t) + QI_2$, $\rho(g_1)^2 = (q-1)\rho(g_1) + qI_2$ and $\rho(g_1)\rho(t)\rho(g_1)\rho(t) = \rho(t)\rho(g_1)\rho(t)\rho(g_1)$. Again, by semisimplicity, the trace can be written as a weighted sum of traces of the matrix algebras

$$\text{tr} = a \text{tr}_{\mathbb{C}} \circ \rho_0 + b \text{tr}_{\mathbb{C}} \circ \rho_1 + c \text{tr}_{\mathbb{C}} \circ \rho_2 + d \text{tr}_{\mathbb{C}} \circ \rho_3 + e \text{tr}_{\mathbb{C}^2} \circ \rho.$$

Now, we look at the images of (basis) elements of $H_2(q, Q)$ under those irreducible representations and construct the system of equations for finding the coefficients a, b, c, d and e :

$H_2(q, Q)$	$\xrightarrow{\sim}$	$\text{End}(\mathbb{C})$	$\oplus$	$\text{End}(\mathbb{C})$	$\oplus$	$\text{End}(\mathbb{C})$	$\oplus$	$\text{End}(\mathbb{C})$	$\oplus$	$\text{End}(\mathbb{C}^2)$
1	$\mapsto$	1	$\oplus$	1	$\oplus$	1	$\oplus$	1	$\oplus$	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$
t	$\mapsto$	Q	$\oplus$	-1	$\oplus$	Q	$\oplus$	-1	$\oplus$	$\begin{pmatrix} Q & Q+q \\ 0 & -1 \end{pmatrix}$
g_1	$\mapsto$	q	$\oplus$	-1	$\oplus$	-1	$\oplus$	q	$\oplus$	$\begin{pmatrix} -1 & 0 \\ 1 & q \end{pmatrix}$
tg	$\mapsto$	Qq	$\oplus$	1	$\oplus$	$-Q$	$\oplus$	$-q$	$\oplus$	$\begin{pmatrix} q & q(Q+q) \\ -1 & -q \end{pmatrix}$
$(tg)^2$	$\mapsto$	Q^2q^2	$\oplus$	1	$\oplus$	Q^2	$\oplus$	q^2	$\oplus$	$\begin{pmatrix} -qQ & 0 \\ 0 & -qQ \end{pmatrix}$

Finally, we get the system of equations for finding a, b, c, d and e :

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 2 \\ Q & -1 & Q & -1 & Q-1 \\ q & -1 & -1 & q & q-1 \\ Qq & 1 & -Q & -q & 0 \\ Q^2q^2 & 1 & Q^2 & q^2 & -2qQ \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \\ e \end{pmatrix} = \begin{pmatrix} 1 \\ s \\ z \\ sz \\ (q-1)(Q-1)sz + (q-1)Qz + qs^2 \end{pmatrix}$$

As we can see, this process is rather tedious but it still works.

5.3 The knot invariant

Using the trace defined in section 5.1, we can do a similar normalization as in S^3 such that both Type II Markov moves affect the trace in the same way, i.e. we want to find $\theta \in \mathbb{C}$ such that for $\alpha \in H_n(q, Q)$, we have

$$\text{tr}(\alpha(\theta g_i)) = \text{tr}(\alpha(\theta g_i)^{-1})$$

Following the same argument as in section 4.4, we get

$$\lambda := \theta^2 = \frac{1 - q + z}{qz}$$

Lemma 5.2. *Thus, if we represent $B_{1,n}$ with $\pi : \sigma_i \mapsto \sqrt{\lambda} g_i \in H_n(q, Q)$, $T \mapsto t \in H_n(q, Q)$, the function*

$$(z\sqrt{\lambda})^{1-n} \text{tr}(\pi(\alpha))$$

only depends on the mixed link $\hat{\alpha}$.

Finally, we get the four-variable invariant given by the following theorem

Theorem 5.3 (Lambropoulou [7]). *The four-variable invariant $X_{\hat{I} \cup L}(q, Q, z, s)$ of the oriented mixed link $\hat{I} \cup L$ that represents an oriented link inside the solid torus M is the function*

$$X_{\hat{I} \cup L}(q, Q, z, s) = (z\sqrt{\lambda})^{1-n} (\sqrt{\lambda})^e \text{tr}(\pi(\alpha))$$

where $\alpha \in B_{1,n}$ such that $\hat{\alpha} = \hat{I} \cup L$ and e is its exponent sum on the σ_i 's and π the representation of $B_{1,n}$ in $H_n(q, Q)$ which maps $T \mapsto t$ and $\sigma_i \mapsto g_i$.

Example 5.2 (unknot, wrapping n times around $\hat{I}$). *The unknot, wrapping n times around $\hat{I}$ is represented by the closure of $T^n \in B_{1,1}$. So, we have*

$$X_{\hat{I} \cup L}(q, Q, z, s) = (z\sqrt{\lambda})^{1-1} (\sqrt{\lambda})^0 \text{tr}(t^n) \stackrel{(17)}{=} \frac{((-1)^{n+1} + Q^n)s + (-1)^n Q + Q^n}{1 + Q}$$

6 Conclusion

In this project, we followed Jones' approach in finding a link invariant by looking at the braid representation of links in S^3 [6]. This braid group is the Artin group of the Coxeter group of A_{n-1} -type, i.e. the Symmetric group S_n . The Markov moves give equivalence classes of braids corresponding to isotopy classes of links. By looking at the Hecke algebra of the braid group, we can construct a trace function, which is compatible with Markov moves in such a way that normalizing the trace gives us a link-invariant.

In a second step, we followed Lambropoulou [7] looking at links in a solid torus. This gives us a braid group, which is the Artin group of the Coxeter group of B_n -type. We follow a similar approach as before by looking at its Hecke algebra, defining a trace and normalizing it to find some link invariant.

It would be interesting to look at braid groups in other 3-manifolds however this might prove difficult when the underlying Coxeter group is infinite.

Appendix A Computations

A.1 Computation of the invariant for the bowline knot

To simplify, let us first introduce some notations: For $k \neq 1$, let a_k, b_k be such that $g_i^k = a_k g_i + b_k$ and define $a_1 := a_{-1}, b_1 := b_{-1}$. More concretely, for our computation, we have:

$$\begin{aligned} g_i^{-1} &= \underbrace{\frac{1}{q}}_{a_1} g_i + \underbrace{\frac{1-q}{q}}_{b_1} \\ g_i^2 &= \underbrace{(q-1)}_{a_2} g_i + \underbrace{q}_{b_2} \\ g_i^3 &= (q-1)g_i^2 + qg_i \\ &= (q-1)((q-1)g_i + q) + qg_i \\ &= \underbrace{(q^2 - q + 1)}_{a_3} g_i + \underbrace{q(q-1)}_{b_3} \end{aligned}$$

Now, we can compute the trace of the braid representation from equation (12):

$$\begin{aligned} \text{tr}(g_1 g_2^{-1} g_1^3 g_2^{-1}) &= \text{tr}(g_1(a_1 g_2 + b_1)(a_3 g_1 + b_3)(a_1 g_2 + b_1)) \\ &= a_1 a_3 a_1 \text{tr}((g_1 g_2 g_1) g_2) + a_1 a_3 b_1 \text{tr}((g_1 g_2) g_1) + a_1 b_3 a_1 \text{tr}(g_1 g_2 g_2) + a_1 b_3 b_1 \text{tr}(g_1 g_2) \\ &\quad + b_1 a_3 a_1 \text{tr}(g_1 g_1 g_2) + b_1 a_3 b_1 \text{tr}(g_1 g_1) + b_1 b_3 a_1 \text{tr}(g_1 g_2) + b_1 b_3 b_1 \text{tr}(g_1) \\ &= a_1^2 a_3 \text{tr}(g_2 g_1 (a_2 g_2 + b_2)) + a_1 a_3 b_1 \text{tr}((a_2 g_1 + b_2) g_2) + a_1^2 b_3 \text{tr}(g_1 (a_2 g_2 + b_2)) + a_1 b_1 b_3 \text{tr}(g_1 g_2) \\ &\quad + a_1 a_3 b_1 \text{tr}((a_2 g_1 + b_2) g_2) + a_3 b_1^2 \text{tr}(a_2 g_1 + b_2) + a_1 b_1 b_3 \text{tr}(g_1 g_2) + b_1^2 b_3 \text{tr}(g_1) \\ &= a_1^2 a_2 a_3 \text{tr}(g_2 (g_1 g_2)) + a_1^2 a_3 b_2 \text{tr}(g_2 g_1) + a_1 a_2 a_3 b_1 \text{tr}(g_1 g_2) + a_1 a_3 b_1 b_2 \text{tr}(g_2) \\ &\quad + a_1^2 a_2 b_3 \text{tr}(g_1 g_2) + a_1^2 b_3 b_2 \text{tr}(g_1) + a_1 b_1 b_3 \text{tr}(g_1 g_2) \\ &\quad + a_1 a_2 a_3 b_1 \text{tr}(g_1 g_2) + a_1 a_3 b_1 b_2 \text{tr}(g_2) + a_2 a_3 b_1^2 \text{tr}(g_1) + a_3 b_1^2 b_2 + a_1 b_1 b_3 \text{tr}(g_1 g_2) + b_1^2 b_3 \text{tr}(g_1) \\ &= \underline{a_1^2 a_2^2 a_3 \text{tr}(g_1 g_2)} + \underline{a_1^2 a_2 a_3 b_2 \text{tr}(g_1)} + \underline{a_1^2 a_3 b_2 \text{tr}(g_1 g_2)} + \underline{a_1 a_2 a_3 b_1 \text{tr}(g_1 g_2)} + \underline{a_1 a_3 b_1 b_2 \text{tr}(g_2)} \\ &\quad + \underline{a_1^2 a_2 b_3 \text{tr}(g_1 g_2)} + \underline{a_1^2 b_3 b_2 \text{tr}(g_1)} + \underline{a_1 b_1 b_3 \text{tr}(g_1 g_2)} \\ &\quad + \underline{a_1 a_2 a_3 b_1 \text{tr}(g_1 g_2)} + \underline{a_1 a_3 b_1 b_2 \text{tr}(g_2)} + \underline{a_2 a_3 b_1^2 \text{tr}(g_1)} + \underline{a_3 b_1^2 b_2} + \underline{a_1 b_1 b_3 \text{tr}(g_1 g_2)} + \underline{b_1^2 b_3 \text{tr}(g_1)} \\ &= (a_1^2 a_2^2 a_3 + a_1^2 a_3 b_2 + a_1 a_2 a_3 b_1 + a_1^2 a_2 b_3 + a_1 b_1 b_3 + a_1 a_2 a_3 b_1 + a_1 b_1 b_3) z^2 \\ &\quad + (a_1^2 a_2 a_3 b_2 + a_1 a_3 b_1 b_2 + a_1^2 b_3 b_2 + a_1 a_3 b_1 b_2 + a_2 a_3 b_1^2 + b_1^2 b_3) z \\ &\quad + a_3 b_1^2 b_2 \\ &= \left(\left(\frac{1}{q} \right)^2 (q-1)^2 (q^2 - q + 1) + \left(\frac{1}{q} \right)^2 (q^2 - q + 1) q + \frac{1}{q} (q-1) (q^2 - q + 1) \frac{1-q}{q} \right. \\ &\quad \left. + \left(\frac{1}{q} \right)^2 (q-1) q (q-1) + \frac{1}{q} \frac{1-q}{q} q (q-1) + \frac{1}{q} (q-1) (q^2 - q + 1) \frac{1-q}{q} + \frac{1}{q} \frac{1-q}{q} q (q-1) \right) z^2 \\ &\quad + \left(\left(\frac{1}{q} \right)^2 (q-1) (q^2 - q + 1) q + \frac{1}{q} (q^2 - q + 1) \frac{1-q}{q} q + \left(\frac{1}{q} \right)^2 q (q-1) q \right. \\ &\quad \left. + \frac{1}{q} (q^2 - q + 1) \frac{1-q}{q} q + (q-1) (q^2 - q + 1) \left(\frac{1-q}{q} \right) + \left(\frac{1-q}{q} \right) q (q-1) \right) z \\ &\quad + (q^2 - q + 1) \left(\frac{1-q}{q} \right) q \end{aligned}$$

A.2 Computation of the recurrence relation for braid group generators

For some computations of Ocneanu's trace, it is useful to have a formula for the generators of the braid group to some power g_i^k for some $k \in \mathbb{N}$. We do this using the recurrence relation equation (5). This gives us the following system of linear equations (for $k \geq 1$):

$$\begin{pmatrix} g_i^{k+1} \\ g_i^k \end{pmatrix} = \underbrace{\begin{pmatrix} q-1 & q \\ 1 & 0 \end{pmatrix}}_{=:A} \begin{pmatrix} g_i \\ 1 \end{pmatrix} \quad (16)$$

Note that A has eigenvalues q with eigenvector $\begin{pmatrix} q & 1 \end{pmatrix}^T$ and -1 with eigenvector $\begin{pmatrix} 1 & -1 \end{pmatrix}^T$. Now, let

$$P := \begin{pmatrix} 1 & q \\ -1 & 1 \end{pmatrix} \quad P^{-1} = \frac{1}{1+q} \begin{pmatrix} 1 & -q \\ 1 & 1 \end{pmatrix}$$

Then, we can diagonalize A like the following:

$$P^{-1}AP = \begin{pmatrix} -1 & 0 \\ 0 & q \end{pmatrix}.$$

Then

$$\begin{aligned} A^k &= P \begin{pmatrix} -1 & 0 \\ 0 & q \end{pmatrix}^k P^{-1} \\ &= \begin{pmatrix} 1 & q \\ -1 & 1 \end{pmatrix} \begin{pmatrix} (-1)^k & 0 \\ 0 & q^k \end{pmatrix} \frac{1}{1+q} \begin{pmatrix} 1 & -q \\ 1 & 1 \end{pmatrix} \\ &= \frac{1}{1+q} \begin{pmatrix} (-1)^k & q^{k+1} \\ (-1)^{k+1} & q^k \end{pmatrix} \begin{pmatrix} 1 & -q \\ 1 & 1 \end{pmatrix} \\ &= \frac{1}{1+q} \begin{pmatrix} (-1)^k + q^{k+1} & (-1)^{k+1}q + q^{k+1} \\ (-1)^{k+1} + q^k & (-1)^kq + q^k \end{pmatrix} \end{aligned}$$

So, finally, we get for $k \geq 1$

$$g_i^k = \frac{\begin{pmatrix} (-1)^{k+1} + q^k \end{pmatrix} g_i + (-1)^k q + q^k}{1+q} \quad (17)$$

As a sanity check, plugging in $k = -1$, $k = 1$, $k = 2$ and $k = 3$, we indeed get the results also calculated explicitly in section A.1.

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